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SEDIMENT TRANSPORT BY TURBULENT FLOW ABOVE A BOTTOM SUBJECT TO EROSION

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The theory of the motion of suspended particles in a turbulent flow at low concentration is presented in [1, 2]. In [3] it is proposed that Coulombic dry friction between the solid particles moving in the liquid be taken into account. In [4-7] the motion of a mixture of a liquid and solid particles is investigated with the help of a rheological relation in the form of a combination of dry friction for the solid phase and viscous friction for the liquid phase. In [4] one-dimensional turbulent flow above an even bottom is considered. In [5-7] the motion is studied in a general formulation with an arbitrary bottom relief and an expression is derived for the sediment flow rate. In [4-7] the particle concentration in the layer of sediment at the bottom is assumed to be constant.

In the present paper we propose, on the basis of the results enumerated above, a model of the medium which gives a continuous description of the motion of the mixture over the entire thickness of the flow, starting from the eroding bottom surface with the limiting particle concentration. Far from the bottom surface, where the concentration is low, the equations convert into the equations of motion derived in [1, 2] for suspended particles in a turbulent flow. The main result is an analytic expression for the sediment flow rate in a turbulent flow for the general three-dimensional problem. The theory does not require the introduction of unknown empirical parameters.

1. Assumptions. We consider the turbulent flow of a heavy incompressible liquid with solid particles in the region $\xi(x, y) < z < \eta$, where x, y, and z is a Cartesian coordinate system whose z-axis is oriented vertically, the equation of the free surface is $z = \eta$, and the equation of the bottom surface is $z = \xi(x, y)$. A stationary granulated uniform medium occupies the region $z < \xi(x, y)$. Mass transfer occurs at the interface $z = \xi(x, y)$. The density of solid particles ρ_p is higher than the density of the liquid ρ_w.

It is assumed that the main mass of the particles moves in a bottom layer of thickness of the order a, much less than the depth $h = \eta - \xi$. The characteristic horizontal since L

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of the flow is significantly greater than the depth. Thus the following inequalities are satisfied:

$$a \ll h \ll L. \tag{1.1}$$

The following consequences can be derived from the main assumption (1.1).

1. The tangential stresses on vertical areas and the acceleration of the liquid along the vertical direction are negligibly small. The pressure distribution is hydrostatic. If the orthogonal coordinates X, Y, Z (where Z is the distance along the normal upwards from the bottom surface) are introduced, then the pressure in the bottom layer at the level Z can be written in the form

$$p = p_a + \rho_w g(\eta - \xi - Z) + g(\rho_p - \rho_w)(a_\infty - a(Z)); \tag{1.2}$$

$$a(Z) = \int_0^Z c(Z)\,dZ, \quad a_\infty = a(\infty), \tag{1.3}$$

where p_a is the atmospheric pressure on the surface of the liquid; the second term is the weight of a column of the pure liquid; a(Z) is the effective thickness of the layer of particles; and, c is the volume concentration of particles in the liquid.

2. The change in the thickness a of the layer in any direction is much smaller than the change in the elevations of the bottom:

$$|\nabla a| \ll |\nabla \xi|, \ \nabla = (\partial/\partial X, \partial/\partial Y). \tag{1.4}$$

For this reason, the pressure gradient in the mixture is equal to the pressure gradient in the pure liquid. In what follows we confine our attention to flows whose Froude number is much less than one, where $|\nabla\eta| \ll |\nabla\xi|$, and for this reason

$$\nabla p = -\rho_w g \nabla \xi. \tag{1.5}$$

3. The angle γ between the vertical direction and the normal to the bottom surface is small. In what follows we take into account first-order infinitesimals in γ, in particular, $\cos\gamma \approx 1$.

4. The diffusion flux through a vertical area is negligibly small.

2. Diffusion Equation. Let u_X and u_Y be the components of the particle velocity parallel to the bottom surface and $u_Z = -w$ the settling velocity of the particles. The particle flux vector $\mathbf{j}$ consists of convective and diffusive fluxes, i.e.,

$$j_X = cu_X, \ j_Y = cu_Y, \ j_Z = -cw - \varepsilon_s \partial c/\partial Z. \tag{2.1}$$

Here ε_s is the turbulent diffusion coefficient, which in the bottom layer is of the order of $\varepsilon_s \sim v_0 a$ (v_0 is the characteristic velocity at the top boundary of the layer).

We write the diffusion equation

$$\partial c/\partial t + \operatorname{div} \mathbf{j} = 0 \tag{2.2}$$

and estimate its terms:

$$\frac{\partial c}{\partial t} + \frac{\partial c u_X}{\partial X} + \frac{\partial c u_Y}{\partial Y} \sim \frac{c v_0}{L}, \quad \frac{\partial}{\partial Z}\varepsilon_s\frac{\partial c}{\partial Z} \sim \frac{c v_0}{a}.$$

It follows from estimates that for $a/L \ll 1$ the diffusion equation (2.2) can be written in the form

$$\frac{\partial}{\partial Z}\left(cw + \varepsilon_s \frac{\partial c}{\partial Z}\right) = 0.$$

Away from the bottom layer the concentration c approaches zero. For this reason, after integration, we have

$$cw + \varepsilon_s \partial c/\partial Z = 0. \tag{2.3}$$

3. Equations of Motion. We project the equation of motion of the bottom layer of the mixture on directions parallel to the bottom:

$$\rho d\mathbf{v}/dt = -\nabla p + \partial\tau/\partial Z - \rho g \nabla \xi,$$
$$\rho = \rho_p c + \rho_w(1 - c) = \rho_w(1 + c(s - 1)), \ s = \rho_p/\rho_w \tag{3.1}$$

where ρ is the density of the mixture and v is the velocity of the mixture.

The tangential stress in the mixture on sections parallel to the bottom consists of the stresses τ_w of the turbulent motion and the friction between the particles τ_f:

$$\tau = \tau_w + \tau_f \tag{3.2}$$

We introduce the characteristic tangential stress at the bottom: $\tau_0 \sim \lambda\rho_w v_0^2/2$.

The coefficient λ is expressed in terms of Chézy's number c_C as $\lambda = 2g/c_C^2$. For sand channels c_C ranges from 30 to 50 $m^{0.5}/sec$. Hence we obtain the required estimate $\tau_0 \sim 0.01 \times \rho_w v_0^2/2$. The values of τ_w and τ_f within the bottom layer range from 0 to τ_0. From the estimates

$$\left|\rho\frac{d\mathbf{v}}{dt}\right| \sim \frac{\rho v_0^2}{L},\quad \frac{\partial\tau_f}{\partial Z} \sim \frac{\partial\tau_w}{\partial Z} \sim \frac{0{,}01\rho_w v_0^2}{a}$$

it follows that acceleration can be neglected for $aL \ll 0.01$. Then, with the help of Eq. (1.5), Eq. (3.1) can be put into the form

$$\partial\tau/\partial Z = c(\rho_p - \rho_w)g\nabla\xi. \tag{3.3}$$

4. Rheology. We assume that τ_f and τ_w are directed along the vector $\partial\mathbf{v}/\partial Z$. Then

$$\boldsymbol{\tau} = \tau\frac{\partial\mathbf{v}}{\partial Z}\Big/\left|\frac{\partial\mathbf{v}}{\partial Z}\right|,\quad \tau = \tau_w + \tau_f. \tag{4.1}$$

The tangential stress τ_w is determined by the turbulent motion of the medium and is found from Prandtl's law [8]

$$\tau_w = (\varkappa Z|\partial\mathbf{v}/\partial Z|)^2\rho_w E(c). \tag{4.2}$$

The function E(c) determines the magnitude of the contributions to the turbulent friction of the liquid and the solid phases. In the limiting case, when the solid phase phase makes no contribution $E(c) = 1 - c$, and when the contributions of the solid and liquid phases are proportional to the mass concentrations, $E(c) = 1 + (s - 1)c$, as assumed in [4]. In [7] the intermediate case ($E = 1$) is assumed.

The tangential stress τ_f is determined from Coulomb's law of dry friction

$$\tau_f = p_s\,\mathrm{tg}\,\varphi,\ p_s = g(\rho_p - \rho_w)(a_\infty - a), \tag{4.3}$$

where φ is the angle of internal friction (for sand $\tan\varphi \approx 0.5$); p_s is the additional pressure in the mixture.

Thus Eq. (4.3) can be represented in the form

$$\tau_f = A(a_\infty - a),\ A = (\rho_p - \rho_w)g\,\mathrm{tg}\,\varphi. \tag{4.4}$$

For the one-dimensional case the proposed rheology agrees with that presented in [4].

5. Boundary Conditions. The velocity and tangential stress are continuous at the interface $Z = 0$. Hence we obtain

$$Z = 0:\ \mathbf{v} = 0,\ \tau_w = 0. \tag{5.1}$$

At the top boundary of the particle layer we assume that the tangential stress $\boldsymbol{\tau} = \mathbf{T}$, is given and the concentration $c = 0$. These conditions can be written as

$$Z \to \infty:\ c \to 0,\ \boldsymbol{\tau} \to \mathbf{T}. \tag{5.2}$$

The limiting values in the conditions (5.2) are reached for $Z \sim a_0$, and for this reason $\mathbf{T}$ equals, to within infinitesimals of order a_0/h, the tangential stress at the bottom in the absence of solid particles in the flow and $\mathbf{T}$ can be found from the relations which are well-known in hydrodynamics.

6. Formulation of the Problem for One-Dimensional Motion. We now write the diffusion equation (2.3), the equation of motion (3.3), and the rheological relation (4.1):

$$\partial\tau/\partial Z = cA\Gamma,\ \Gamma = (\partial\xi/\partial X)/\mathrm{tg}\,\varphi; \tag{6.1}$$

$$cw + \varepsilon_s\partial c/\partial Z = 0; \tag{6.2}$$

$$\tau = A(a_\infty - a) + \tau_w,\ \tau_w = \rho_w\varepsilon\partial v/\partial Z; \tag{6.3}$$

$$\varepsilon_s = k\varepsilon,\ \varepsilon = (\varkappa Z)^2\partial v/\partial Z,\ \varkappa \approx 0{,}4. \tag{6.4}$$

The diffusion coefficient ε_s is proportional to the coefficient of turbulent viscosity ε, and k is of the order of unity [9].

Thus in order to calculate $c(Z)$ and $v(Z)$ the system of equations (6.1)-(.4) must be solved with the boundary conditions (5.1) and (5.2).

6a. Determination of the Tangential Stresses and Thickness of the Moving Particle Layer. We integrate Eq. (6.1) along Z and obtain, with the help of Eq. (6.3),

$$\tau = A(a_\infty - a) + \tau_w = T + A\Gamma(a - a_\infty). \tag{6.5}$$

It follows from Eq. (6.5) that τ_w is a linear function of the variable a, and in addition $\tau_w(0) = 0$, $\tau_w(a_\infty) = T$. Hence

$$\tau_w = T(a/a_\infty). \tag{6.6}$$

On the basis of Prandtl's law (6.3) and (6.4), we obtain from Eq. (6.6)

$$\varkappa Z \frac{\partial v}{\partial Z} = u_* \left(\frac{a}{a_\infty E}\right)^{1/2}, \quad u_* = \left(\frac{T}{\rho_w}\right)^{1/2}; \tag{6.7}$$

$$\varepsilon = \varkappa Z u_* \left(\frac{a}{a_\infty E}\right)^{1/2} \tag{6.8}$$

where u_* is the dynamic viscosity. From Eq. (6.5) and the conditions that at the bottom $a = 0$ and $\tau_w = 0$, we find

$$a_\infty = \frac{T}{A(1+\Gamma)} = \frac{u_*^2}{(s-1)\, g \operatorname{tg} \varphi (1+\Gamma)}, \quad s = \frac{\rho_p}{\rho_w}. \tag{6.9}$$

6b. Distribution of the Particle Concentration. We now substitute into the diffusion equation (6.2) the expression (6.8) for the turbulent diffusion ε_s. Then we find for the concentration $c(Z)$ and the thickness $a(Z)$ of the layer

$$\alpha c + \left(\frac{a}{a_\infty E}\right)^{1/2} Z \frac{dc}{dZ} = 0, \quad \frac{da}{dZ} = c; \tag{6.10}$$

$$\alpha = w/u_* \varkappa k. \tag{6.11}$$

There exists a limiting concentration c_{sat} (saturation concentration) for the moving mixture. For this reason, Eq. (6.10) is valid in the region $\delta \leqslant Z$, where $c \leqslant c_{sat}$.

In the layer $0 \leqslant Z \leqslant \delta$ the concentration is constant $c = c_{sat}$, and the value of a increases linearly with Z. Thus in the region $0 \leqslant Z \leqslant \delta$ we have

$$c = c_{sat}\, a = c_{sat} Z. \tag{6.12}$$

In the region $\delta \leqslant Z$ the concentration distribution must be determined from the system of equations (6.10) with the boundary conditions

$$\begin{gathered} Z = \delta\colon a = a_0 = c_{sat}\,\delta,\ c = c_{sat}, \\ Z = \infty\colon a = a_\infty,\ c = 0. \end{gathered} \tag{6.13}$$

For low concentration ($c \ll 1$) we can set $E = 1$. Then the first equation in Eqs. (6.10), taking into account Eq. (6.13), has the following integral:

$$2\alpha (a_\infty a)^{1/2} + Z da/dZ - a = (2\alpha - 1) a_\infty. \tag{6.14}$$

From Eq. (6.14) and the conditions (6.13) with $Z = \delta$ we find

$$c_{sat}\,\delta / a_\infty = a_0/a_\infty = (1 - 1/2\alpha)^2. \tag{6.15}$$

With the help of the substitution

$$a/a_\infty = (1 - \psi)^2 \tag{6.16}$$

Eq. (6.14) can be put into the form

$$-2(1-\psi) d\psi / [\psi(2\alpha - 2 + \psi)] = dZ/Z, \quad \psi(\delta) = 1/2\alpha, \tag{6.17}$$

which can be integrated, giving the solution of the boundary-value problem (6.10) and (6.13):

$$\begin{gathered} Z = \frac{a_\infty (2\alpha - 2 + \psi)^{(2\alpha-1)/(\alpha-1)}}{c_{sat} (2\alpha - 1)^{2\alpha/(\alpha-1)} \psi^{1/(1-\alpha)}}, \\ c = 2a_\infty (\psi - 1) \frac{d\psi}{dZ} = c_{sat} \left(\frac{(2\alpha-1)^2 \psi}{2\alpha - 2 + \psi}\right)^{\alpha/(\alpha-1)}. \end{gathered} \tag{6.18}$$

The asymptotic concentration distribution for $Z \gg \delta$ is found from the solution (6.18), substituting Eq. (6.15):

$$c = c_{\text{sat}} \left(\frac{\delta}{Z}\right)^{\alpha} \left(1 - \frac{1}{(2\alpha - 1)^2}\right)^{2\alpha}. \tag{6.19}$$

It can be shown that for sufficiently large α the solution can be represented, with the relative error $1/(2\alpha - 1)$, by the approximate formula

$$c = c_{\text{sat}} (\delta/Z)^{\alpha}. \tag{6.20}$$

The results (6.18)-(6.20) obtained above are valid for $\alpha > 1$. For $\alpha = 1$ the solution has the form

$$Z/\delta = 4\psi^2 e^{2/\psi - 4}, \; c = c_{\text{sat}} e^{4 - 2/\psi}, \; \delta = a_\infty/4c_{\text{sat}}. \tag{6.21}$$

In this case, for $Z \gg \delta$ we have the asymptotic expansion

$$c = c_{\text{sat}}\, 4\psi^2 \frac{\delta}{Z} \approx \frac{4c_{\text{sat}}}{(\ln (Z/\delta))^2} \frac{\delta}{Z}. \tag{6.22}$$

The boundary value problem (6.10) and (6.13) does not have a solution for $\alpha < 1$.

6c. Sediment Flow Rate. We now find the sediment flow rate under the assumption that the particles move with the velocity of the mixture. Integrating by parts and using the formula (6.7) we obtain

$$G = \int_0^\infty \rho_p c v\, dZ = \int_0^\infty \rho_p v \frac{\partial}{\partial Z} (a - a_\infty)\, dZ = \rho_p \int_0^\infty (a_\infty - a) \frac{\partial v}{\partial Z} dZ = \frac{\rho_p u_*}{\varkappa} \int_0^\infty (a_\infty - a) \left(\frac{a}{a_\infty}\right)^{1/2} dZ.$$

On the basis of Eqs. (6.7) and (6.16) we write

$$G = \frac{\rho_p a_\infty u_*}{\varkappa} I, \quad I = \int_0^\infty \psi (2 - \psi)(1 - \psi) \frac{dZ}{Z}. \tag{6.23}$$

As one can see from Eq. (6.23), the flow rate depends on the slope Γ via a_∞. Substituting the expression (6.9) for a_∞, we have

$$G = \frac{G_0}{1 + \Gamma}, \quad G_0 = \frac{\rho_p u_*^3 I}{\varkappa (s - 1) g \operatorname{tg} \varphi}. \tag{6.24}$$

According to Eqs. (6.12), (6.16), and (6.17), we find

$$\frac{dZ}{Z} = \begin{cases} -\dfrac{2d\psi}{1 - \psi}, & 0 \leqslant Z \leqslant \delta, \; \dfrac{1}{2\alpha} \leqslant \psi \leqslant 1, \\ -\dfrac{2(1 - \psi)\, d\psi}{\psi (2\alpha - 2 + \psi)}, & \delta < Z, \; 0 \leqslant \psi \leqslant \dfrac{1}{2\alpha}. \end{cases} \tag{6.25}$$

The integral (6.23) can be easily calculated with the help of the substitution (6.25). We now give the asymptotic expansion in the small parameter $1/2\alpha$:

$$I = 4/3 + 1/2\alpha^2 + \ldots, \; 1/2\alpha \ll 1. \tag{6.26}$$

In the approximation (6.26) the bottom layer of thickness δ with constant concentration c_{sat} makes the main contribution to the sediment flow rate ($I \approx 4/3$). For $Z \gg \delta$ the concentration of suspended sediment decreases rapidly according to the power law (6.20) and makes a relatively small contribution ($\sim 1/\alpha^2$).

For $0 < \alpha - 1 \ll 1$ the following asymptotic expression can be derived:

$$I \approx 4 \ln \frac{1}{\alpha - 1}, \quad \alpha - 1 \ll 1. \tag{6.27}$$

In this case, which corresponds to high flow rate, the relative contribution of the suspended part of the sediment to the flow rate is much greater than the contribution of the bottom layer of thickness δ.

6d. Taking into Account the Relative Velocity of the Particles. We now project the equation of balance of the forces acting on the dispersed phase per unit volume of the mixture onto a plane tangent to the bottom surface:

$$F_R + F_C + F_g = 0. \tag{6.28}$$

The first term is the resistance force of the particles in the liquid proportional to the squared relative velocity of the particles v_{rel}. It can be represented in the form

$$F_{\mathrm{R}} = \frac{c}{V_{\mathrm{p}}} f, \quad f = -(\rho_{\mathrm{p}} - \rho_{\mathrm{w}}) g V_{\mathrm{p}} \frac{|v_{\mathrm{rel}}| v_{\mathrm{rel}}}{w^2} \Rightarrow$$
$$F_{\mathrm{R}} = -Ac \frac{|v_{\mathrm{rel}}| v_{\mathrm{rel}}}{w^2 \operatorname{tg} \varphi} \tag{6.29}$$

where V_p is the volume of an individual particle, c/V_p is the number of particles per unit volume, and f is the resistance force acting on an individual particle. For $|v_{rel}| = |w|$, evidently, f is the weight of the solid particle minus the buoyancy force.

The second term is the Coulomb friction force. With the help of (4.4) we obtain

$$F_{\mathrm{C}} = \frac{\partial \tau_{\mathrm{f}}}{\partial Z} = -Ac. \tag{6.30}$$

Finally, the projection of the force F_g, acting on a particle in the liquid as a result of the presence of the acceleration of gravity, can be represented as

$$F_g = -Ac\Gamma. \tag{6.31}$$

Substituting the expressions (6.29)-(6.31) into Eq. (6.28), we find the relative velocity of the particles in the liquid:

$$v_{\mathrm{rel}}^2 = w^2 \operatorname{tg} \varphi (1 + \Gamma), \tag{6.32}$$

and the vector v_{rel} is oriented opposite to the velocity v of the mixture.

Taking into account the relative velocity (6.32), the particle flow rate will change by the amount

$$\Delta G = \rho_{\mathrm{p}} v_0 \int_0^\infty c\,dZ = \rho_{\mathrm{p}} v_{\mathrm{rel}} a_\infty. \tag{6.33}$$

Thus we find with the help of Eqs. (6.23), (6.24), and (6.33) the following expression for the flow rate:

$$G' = G + \Delta G = G_0 \left(1 - \frac{u_{*\mathrm{T}}}{u_*}\right) \frac{1}{1+\Gamma}; \tag{6.34}$$

$$u_{*\mathrm{T}} = \frac{\varkappa}{I} |v_{\mathrm{rel}}| \tag{6.35}$$

where u_{*T} is the dynamic velocity corresponding to the instant at which the particles come into contact. We note that the formulas (6.34) and (6.35) do not contain any unknown empirical constants. The velocity v_p of the solid particles is oriented in the same direction as the velocity v(Z) of the mixture. For $v < |v_{rel}|$ the velocity of the particles is zero. Thus the particle velocity distribution over depth is determined as follows:

$$v_{\mathrm{p}} = v(Z) + v_{\mathrm{rel}}, Z \geqslant Z_0, \; v_{\mathrm{p}} = 0, \; 0 \leqslant Z \leqslant Z_0 < \delta. \tag{6.36}$$

The level Z_0, separating the stationary layer of particles from the moving layer, is found from the relations

$$-v_{\mathrm{rel}} = w \sqrt{\operatorname{tg} \varphi} \left(1 + \frac{\Gamma}{2}\right) = v(Z_0) = \int_0^{Z_0} \frac{\partial v}{\partial Z} dZ.$$

With the help of Eqs. (6.7) and (6.12) the last integral can be calculated in the form

$$v(Z_0) = \frac{u_*}{\varkappa} \int_0^{Z_0} \left(\frac{a}{a_\infty}\right)^{1/2} \frac{dZ}{Z} = 2 \frac{u_*}{\varkappa} \left(\frac{c_{\mathrm{sat}} Z_0}{a_\infty}\right)^{1/2},$$

whence $c_{\mathrm{sat}} Z_0 / a_\infty = (v_0 \varkappa / 2u_*)^2$. According to Eqs. (6.32) and (6.35), we obtain

$$\frac{c_{\mathrm{sat}} Z_0}{a_\infty} = \left(\frac{I u_{*\mathrm{T}}}{2u_*}\right)^2 = \operatorname{tg} \varphi \frac{\varkappa^2 k^2}{4} \alpha^2 (1 + \Gamma). \tag{6.37}$$

The flow rate (6.33) must be calculated more accurately taking into account the distribution (6.36), in which for $0 \leqslant Z \leqslant Z_0$ the solid particles are at rest. Using Eqs. (6.37) and (6.35), we estimate the error in the formula (6.33) as

$$\rho_{\mathrm{p}} c_{\mathrm{sat}} \int_0^{Z_0} (v + v_{\mathrm{rel}}) \frac{dZ}{\Delta G} = \frac{2u_*}{3\varkappa |v_{\mathrm{rel}}|} \left(\frac{c_{\mathrm{sat}} Z_0}{a_\infty}\right)^{3/2} = \frac{\varkappa^2 v_{\mathrm{rel}}^2}{12 u_*^2} = \frac{I^2}{12} \left(\frac{u_{*\mathrm{T}}}{u_*}\right)^2.$$

7. Solution of the General Problem. From the diffusion equation (2.3), the equation of motion (3.3), and the rheological relation (4.1) we obtain, by analogy to the one-dimensional case, instead of (6.5)

$$\boldsymbol{\tau} = \mathbf{R},\ \mathbf{R} = \mathbf{T} + A\boldsymbol{\Gamma}(a - a_\infty),\ \boldsymbol{\Gamma} = \nabla\xi \operatorname{ctg} \varphi. \tag{7.1}$$

Within the adopted assumptions $\Gamma \ll 1$

$$R = T + A\Gamma_\tau(a - a_\infty) \tag{7.2}$$

(Γ_T is the projection of the vector $\boldsymbol{\Gamma}$ on the vector $\mathbf{T}$). From the rheological relation (4.1)-(4.4) and Eq. (7.1) we find

$$R = \tau_f + \tau_w = A(a_\infty - a) + \tau_w. \tag{7.3}$$

Since $\tau_w = 0$ at $Z = 0$, it follows from Eqs. (7.2) and (7.3) that

$$a_\infty = T/(A(1 + \Gamma_\tau)). \tag{7.4}$$

By analogy to the one-dimensional problem, we write

$$\tau_w = T\frac{a}{a_\infty},\quad \varkappa Z\left|\frac{\partial \mathbf{v}}{\partial Z}\right| = u_*\left(\frac{a}{a_\infty E}\right)^{1/2}. \tag{7.5}$$

Repeating all arguments in Sec. 6c, we find that Eqs. (6.10)-(6.16) are also valid for the two-dimensional case.

Since the vectors $\boldsymbol{\tau}$, $\mathbf{R}$, and $\partial\mathbf{u}/\partial Z$ are collinear, it follows from Eq. (7.5) that

$$\frac{\partial \mathbf{v}}{\partial Z} = \frac{u_*}{\varkappa Z}\left(\frac{a}{a_\infty E}\right)^{1/2}\frac{\mathbf{R}}{R}. \tag{7.6}$$

From Eq. (7.1) we find

$$\frac{\mathbf{R}}{R} = \frac{\mathbf{T}}{T}(1 + \Gamma_\tau(2\psi - \psi^2)) - \boldsymbol{\Gamma}(2\psi - \psi^2). \tag{7.7}$$

We determine the sediment flow rate, by analogy to the one-dimensional case, under the assumption that the particles move with the velocity of the mixture:

$$\mathbf{G} = \rho_p\int_0^\infty \mathbf{v}c\,dZ = \rho_p a_\infty\int_0^\infty (2\psi - \psi^2)\frac{\partial \mathbf{v}}{\partial Z}\,dZ. \tag{7.8}$$

Substituting the expressions (7.6) and (7.7) into Eq. (7.8), we find the final expression for the flow rate:

$$\mathbf{G} = G_0\left((1 - \Gamma_\tau(1 - B))\frac{\mathbf{T}}{T} - B\boldsymbol{\Gamma}\right),\quad I_2 = \int_0^\infty (1 - \psi)\,\psi^2(2 - \psi)^2\frac{dZ}{Z},\ B = I_2/I. \tag{7.9}$$

The integral I is calculated in Sec. 6c. The integral I_2 can be found similarly, and as a result we obtain the asymptotic formulas

$$I_2 = \frac{16}{15} + \frac{1}{6\alpha^3},\quad B \approx \frac{4}{5},\quad \alpha \gg 1,\quad I_2 = 8\ln\frac{1}{\alpha - 1},\ B \approx 2,\ 0 < (\alpha - 1) \ll 1. \tag{7.10}$$

The relative velocity of the particles can be taken into account similarly to the discussions in Sec. 6d. Thus we obtain

$$\mathbf{G} = G_0\left(1 - \frac{u_{*\tau}}{u_*}\right)\left[\left(1 - \Gamma_\tau(1 - B)\right)\frac{\mathbf{T}}{T} - B\boldsymbol{\Gamma}\right]; \tag{7.11}$$

$$u_{*\tau} = (\varkappa/I)\sqrt{\operatorname{tg}\varphi}\,(1 + \Gamma_\tau/2)\,w. \tag{7.12}$$

8. Comparison with Experiment. We now compare with experimental data the theoretical formulas for the contact velocity of the particles (7.12) and the sediment flow rate (7.11), substituting into them the known values of the parameters ($\varkappa = 0.4$, $\tan\varphi \approx 0.5$). It follows from Eq. (7.12) that on an even bottom the ratio $u_{*\tau}/w \approx 0.2$. In [10] it was established, on the basis of experimental data, that this ratio ranges from 0.18 to 0.25.

As noted in [11], one of the most reliable empirical formulas is [12]

$$G = \frac{8\rho_p}{s - 1}g^{1/2}\left(\frac{u_*^2}{g} - 0{,}047d\right)^{3/2} \tag{8.1}$$

where d is the diameter of the particles. Using for the setting velocity of the particles the formula $w^2 = (s - 1)gd$ [11] and Eq. (7.12), Eq. (8.1) can be put into the form

$$G = \frac{8\rho\ u_*^3}{(s-1)\,g}\left(1-\left(\frac{u_{*T}}{u_*}\right)^2\right)^{3/2}.$$

The range of interest for practical applications is $u_{*T}/u_* < 0.9$, where the flow rates calculated from the formulas (8.1) and (6.34) differ by not more than 20% with $\Gamma = 0$.

The structure of the formulas (7.11) and (7.12) is identical to that obtained previously in [7] neglecting the diffusion of the particles. The values of the coefficients in the formula for the flow rate are close over the entire range of the parameter α, except close to one. Therefore, the check of the formula in [7] for the case of erosion of the banks of a channel also pertains to the present work, and it can be concluded that the formulas for the sediment flow rate on an uneven bottom are in good agreement with experiment.

Finally, the power law obtained for the concentration distribution (6.20) agrees with the known results of [11, 13].

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